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COMMUTATIVE RINGS, ALGEBRAIC TOPOLOGY, GRADED LIE ALGEBRAS AND THE WORK OF Jan-Erik ROOS

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Dedicated to Jan-Erik Roos on his 50-th birthday

The classic book, Homological Algebra, by Cartan and Eilenberg begins in the following way:

'During the last decade the methods of algebraic topology have invaded extensively the domain of pure algebra ... The invasion has occurred on three fronts through the construction of cohomology theories for groups, Lie algebras and associative algebras ... We present here a single cohomology (and also a homology) theory which embodies all three.'

A few lines later the authors mention Hilbert's syzygy theorem as an application; this is one instance of the usefulness of these methods when applied to the study of commutative rings.

A central role in Homological Algebra is played by the functors Tor and Ext. Applied to a ring homomorphism $R \to \mathbf{k}$ ($\mathbf{k}$ a field) they give a graded $\mathbf{k}$-vector space $\mathrm{Tor}^R_*(\mathbf{k},\mathbf{k})$ whose graded dual $E^*_R = \mathrm{Ext}^*_R(\mathbf{k},\mathbf{k})$ is, equipped with the Yoneda product, a graded algebra.

With the introduction of differential homological algebra by Eilenberg and Moore [14] the functor Tor was extended to differential graded algebras (DGA's). They showed, in particular, that for the cochain algebra C^* of a pointed 1-connected CW complex of finite type: $\mathrm{Tor}^{C^*}(\mathbf{k},\mathbf{k}) = H^*(\Omega X;\mathbf{k})$, ΩX being the loop space of X. Dualizing gives a graded algebra $E^{X,\mathbf{k}}_* = H_*(\Omega X;\mathbf{k})$, with multiplication induced from composition of loops.

The parallel between E^*_R and $E^{X,\mathbf{k}}_*$ is even clearer when (on the topological side) we restrict to $\mathbf{k} = \mathbb{Q}$. (Thus we write $E^X_* = E^{X,\mathbb{Q}}_*$). Indeed, Milnor and Moore [31] showed that E^X_* is the universal enveloping algebra UL^X_* of a canonical graded Lie

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algebra L_*^X. Their work (for char $\mathbf{k}=0$) together with André [1] (char $\mathbf{k}>2$) and Sjödin [39] (char $\mathbf{k}=2$) establishes that if R is a commutative noetherian local ring, then E_R^* is also the universal enveloping algebra of a canonical graded Lie algebra L_R^*. Henceforth we shall restrict ourselves to commutative rings R.

A third 'level' of technique was provided by Quillen's homotopical algebra in 1967. Applied by him [32] and subsequently by Sullivan [40] it gave a homotopy theory for graded commutative DGA's over $\mathbb{Q}$ which was equivalent to rational homotopy theory for topological spaces. In particular, Quillen showed that every rational graded Lie algebra $L=\bigoplus_{k>0} L_k$ (dim L_k finite, each k) was of the form $L=L_*^X$ for suitable 1-connected X (of finite type).

Now this is very definitely *not* true if X is required to be finite, and the analogous assertion for local commutative noetherian rings (each L^* has the form L_R^*) is equally *not* true. In fact these two classes of graded Lie algebras turn out to behave in a remarkably similar way, and this fact was recognized even in the 1950's and early 1960's with the following question of Serre and Kaplansky (cf. [38]):

Problem 1. If R is a local commutative noetherian ring with residue field $\mathbf{k}$, and if X is a finite 1-connected CW complex, do the series

$$E_R(z)=\sum_i \dim[\mathrm{Ext}_R^i(\mathbf{k},\mathbf{k})]z^i \quad \text{and} \quad E^X(z)=\sum_i [\dim H_i(\Omega X;\mathbb{Q})]z^i$$

always represent rational functions of z?

More recently, this has been 'generalized' to the following fundamental question:

Problem 2. What are the properties of the graded Lie algebras of the form $L=L_R^*$ or L_*^X (R, X as in Problem 1)? What are the properties of the algebras E_R^*, E_*^X? In particular, what can one say about the series $L_R(z)$, $L^X(z)$, $E_R(z)$ and $E^X(z)$?

On the topological side the first significant contribution was made by Lemaire [26] in the early seventies, starting from a suggestion of Moore. He considered CW complexes obtained from a finite bouquet of spheres $\vee S^{n_\alpha}$ by attaching finitely many cells $e^{n_\beta+1}$ via an attaching map $f:\vee S^{n_\beta}\to\vee S^{n_\alpha}$. (A recent result of Y. Félix and J.C. Thomas [17] shows these are exactly the finite 1-connected CW complexes whose rational Lusternik–Schnirelmann category is two). Although Problem 1 was not resolved, Lemaire did construct such a complex X for which E_*^X was not a finitely generated algebra.

By the early 1970's substantial effort had been directed towards Problem 1 on the algebra side. An important step was taken with Levin's theorem [27] that if $\bar{R}=R/\mathfrak{m}^k$ ($\mathfrak{m}$ the maximal ideal, k sufficiently large), then $E_R(z)^{-1}-E_{\bar{R}}(z)^{-1}$ was indeed a rational function, so that Problem 1 was reduced to that of rings satisfying $\mathfrak{m}^k=0$, some k.

This, then, was the context in which in 1975 Jan-Erik Roos began a research program on the homological properties of local rings, with specific focus on those whose maximal ideal, $\mathfrak{m}$, satisfied $\mathfrak{m}^3=0$. (This was the first non-trivial case for Problem 1.) This program, actively pursued by Roos and his students, and others, has had significant implications in three separate directions; namely

(i) The discovery that (in addition to technique) the two fields - local rings and homotopy theory - shared a common body of non-trivial theorems for which one could sometimes give a single proof.

(ii) The essential role which graded Lie algebras play in both subjects.

(iii) The remarkable properties of the class of local rings R with $\mathfrak{m}^3=0$.

It is interesting to note how developments (i) and (ii) were foreshadowed by Cartan and Eilenberg.

The development (i) can be traced directly to Roos' paper [34] (which reports work done in 1975–76) in which he showed that Problem 1 (for rings with $\mathfrak{m}^3=0$ and residue field $\mathbb{Q}$) and Problem 1 (for CW complexes of dimension four) were equivalent. Thus when in 1979 Anick [3] settled Problem 1 for CW complexes by constructing an example with $E^X(z)$ irrational his space produced a local ring R with $\mathfrak{m}^3=0$ and such that $E_R(z)$ was irrational as well.

Nor even in its inception was the method limited to questions of series. In fact, Roos applied it to Lemaire's example (mentioned earlier) to get a local ring R (with $\mathfrak{m}^3=0$) whose Ext-algebra was not finitely generated.

This idea of transferring theorems, rather than just techniques, between topology and algebra has really been extraordinarily fruitful, and particularly so in the attacks on Problem 2. Thus Avramov [6] was able to transfer from topology a theorem of Félix–Halperin–Thomas [15]. This asserts that for (possibly infinite) X of finite Lusternik–Schnirelmann category, either L_*^X is finite-dimensional or else the numbers $\sum_{i=0}^n \dim E_i^X$ grow exponentially in n. Avramov's theorem (of which a special case is due, independently to Félix–Thomas [16]) is that if R is a local noetherian ring, then either R is a complete intersection or the numbers $\dim(E_R^n)$ grow exponentially in n.

In the other direction a theorem of Avramov and Levin [28] described the algebra E_R^* when R had the form S/I with S artinian Gorenstein and I the socle. (When S was a complete intersection the Poincaré series of S/I had been obtained earlier by Gulliksen [20].) This theorem was transferred to topology (for formal manifolds by Avramov [6], in general by Halperin and Lemaire [21]) where it describes the algebra E_*^X when X is the $(n-1)$-skeleton of a closed n-manifold. This transfer has a non trivial further application: it is used in [22] to prove that a circle cannot act freely on a finite connected sum of Lie groups (of rank ≥ 2).

There are many examples now of this phenomenon (some in this issue) and a general account is available in [6] and [8].

The explicit role of graded Lie algebras (development (ii)) is already clearly present in the theses of Lemaire [26] and Löfwall [29]. In fact, if X arises from an attaching map $f: \bigvee S^{n_\beta} \to \bigvee S^{n_\alpha}$, then one can identify $H_*(\Omega f; \mathbb{Q})$ as a homomorphism between finitely generated tensor algebras $T(W) \to T(V)$, and the quotient algebra $A = T(V)/\langle H_*(\Omega f;\, \mathbb{Q})(W)\rangle$ is the universal enveloping algebra of a finitely presented graded Lie algebra. What Lemaire shows is that (a) all such A's arise in this way and (b) the series $A(z)$ is rationally related to $E^X(z)$.

Analogously, Löfwall considers the subalgebra $A \subset E_R^*$ generated by E_R^1 and obtains a rational relation between the series $A(z)$ and $E_R(z)$ when R satisfies $\mathfrak{m}^3 = 0$. This was also obtained (in a different way) by Roos in [34].

For such R, moreover, A is also the universal enveloping algebra of a finitely presented graded Lie algebra, but now with generators in degree 1 and relations in degree 2. In [34] Roos shows that all such A's occur in this way (this is also implicit in [29]) and coïncide exactly with the A's of Lemaire in the case of four-dimensional complexes. This is, of course, a key step in his theorem.

It also meant that in order to construct rings R ($\mathfrak{m}^3 = 0$) with $E_R(z)$ irrational it was sufficient to construct such a (1–2) presented Lie algebra L for which the series $UL(z)$ was irrational. In [30] Löfwall and Roos give a new technique (i.e. different from that of Anick) for such a construction, thus providing a second way of making a counter example for Problem 1. Their construction depends exclusively on calculations with, and properties of the cohomology of graded Lie algebras.

In the last five years the recognition of the role played in this subject by graded Lie algebras, their enveloping algebras and more general graded associative algebras has (cf. [2], [21], [23], ...) resulted in work directly in these domains. In particular, there is Bøgvad's beautiful theorem [12] that if L is a finitely generated graded Lie algebra of global dimension 2 (over an algebraically closed field), then an element in the centre of L is either a generator or the square of a generator.

The third development, the study of $E_R(z)$ for rings with $\mathfrak{m}^3 = 0$, might have been expected to die with Anick's counter-example. After all, it did appear that the primary interest of the restriction $\mathfrak{m}^3 = 0$ was precisely that it was the simplest place one might hope for such an example.

In fact, an enormous wealth of examples and variety of behaviour have now been exhibited by the study of this class of rings. There is (for instance) the result of Fröberg, Gulliksen and Löfwall [19] which provides a flat family of local artinian algebras with an infinite number of series $E_R(z)$. Analogously, in topology, Anick [4] and Avramov [7] construct finite 1-connected CW complexes X in dimension four with non-trivial p torsion (all primes p) in $H_*(\Omega X;\mathbb{Z})$.

But the most remarkable development is contained in the article of Anick and Gulliksen in this issue [5]. Building on the earlier work of Roos (and Lemaire, Löfwall, ...) and on results of Jacobsson [25] (also in this issue) they show that the series $E_S(z)$ for any local noetherian S is rationally related to a series $E_R(z)$ for some local noetherian R with $\mathfrak{m}^3 = 0$. Moreover, the series $E^X(z)$ (X a finite 1-connected CW complex) are all rationally related to these series (and conversely) and to the series arising from X of dimension four.

Now once the question of rationality was settled negatively, the question arose whether there was some other easily described countable set of series containing this one. This was answered positively by Jacobsson and Stoltenberg-Hansen who proved [24] that all series $E_R(z)$ are primitive recursive.

The problem of the behaviour of the series $E_R(z)$ has also been studied by concentrating on other classes of rings. Thus Fröberg [18] began the study of rings with monomial relations, and Backelin [10] the study of Golod attached rings. In [9] Backelin shows that for rings with monomial relations $E_R(z)$ is rational, and then Backelin and Roos prove [11] the very surprising fact that for such rings the algebra $\mathrm{Ext}_{\mathrm{Ext}_R(\mathbf{k},\mathbf{k})}(\mathbf{k},\mathbf{k})$ is noetherian.

There is, as might be expected, considerable work on Problem 2, old, new or still in progress which has not been mentioned. We should, however, indicate Roos' interest in coherence [33], and in λ-dimension and finitistic global dimension. In [36] and [37], he establishes results which suggest that these may be very useful invariants in the future.

The influence of Jan-Erik Roos on this entire area has been significant. Backelin, Bøgvad, Fröberg, Jacobsson and Löfwall are in fact colleagues, and all of these but Fröberg were his students. His ideas have had a considerable impact with many others, including those of us (Anick, Avramov, Gulliksen, Halperin, Lemaire, Levin...) who have spent time working with him in Stockholm.

Not least among his contributions was the realization that one group of mathematicians in topology, and another in algebra, were working at subjects with so much in common that a profitable scientific collaboration was possible. The energy and initiative which he has shown in realizing this idea (culminating with his conference in 1983 on Algebra, Algebraic Topology and their Interaction) have resulted in active joint research efforts by members of these two groups.

The substantial benefit to the subject (and its researchers) of this collective effort is indicated by the articles in the proceedings of that conference and in this special issue.

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